

# 小さな箱の中の強い力

## — 格子QCDと有限体積効果

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# はじめまして！

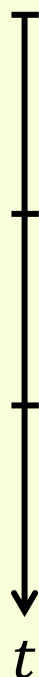
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いま興味のあるもの

- ・有限温度密度QCD
- ・テンソルネットワーク
- ・符号問題

QCDなら割となんでも…

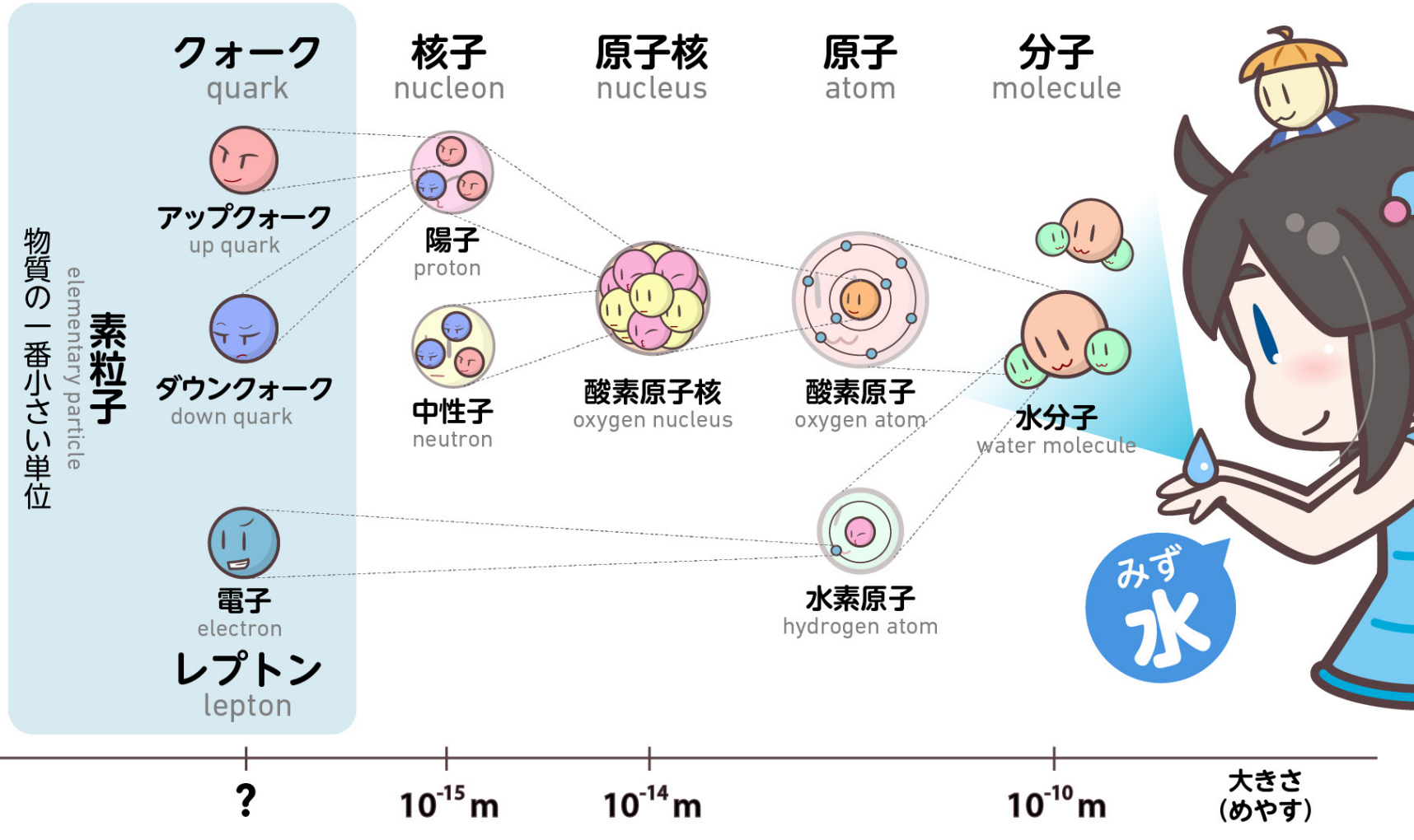
# Introduction



# ミクロな世界へようこそ！

小さい

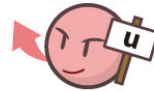
大きい



## 物質粒子 matter (fermions)

第1世代      第2世代      第3世代

クォーク  
quarks



アップ  
クォーク



ダウン  
クォーク



チャーム  
クォーク



ストレンジ  
クォーク



トップ  
クォーク



ボトム  
クォーク

レプトン  
leptons



電子



電子  
ニュートリノ



ミュー粒子



ミュー  
ニュートリノ



タウ粒子



タウ  
ニュートリノ

## ゲージ粒子 gauge bosons

電磁気力  
electromagnetic



光子 (フォトン)

強い力  
strong



グルーオン

弱い力  
weak



ウィークボソン

## ヒッグス粒子 Higgs bosons

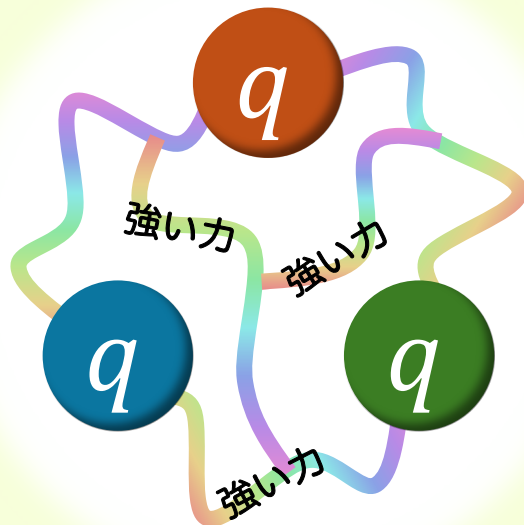


ヒッグス粒子



## 量子色力学 (Quantum Chromodynamics)

- ・ 中身はカラフル (観測できない)
- ・ 外から見ると白色 (ハドロン)  
= 私たちが観測できるもの



今までに見つかっているハドロンは… **300** 種類以上！

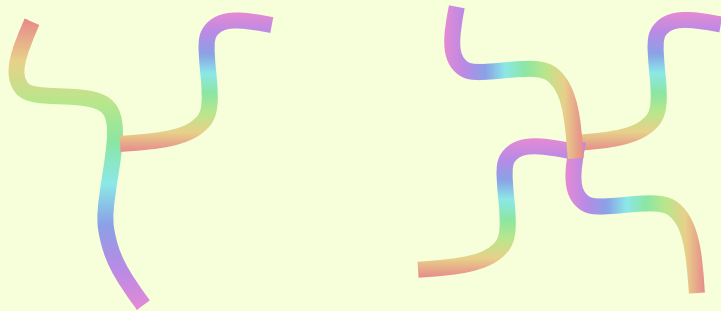
## 量子色力学 (Quantum Chromodynamics)

- ・ 中身はカラフル (観測できない)
- ・ 外から見ると白色 (ハドロン)  
= 私たちが観測できるもの



なぜそうなのか？ … わからない。

グルーオンの自己相互作用

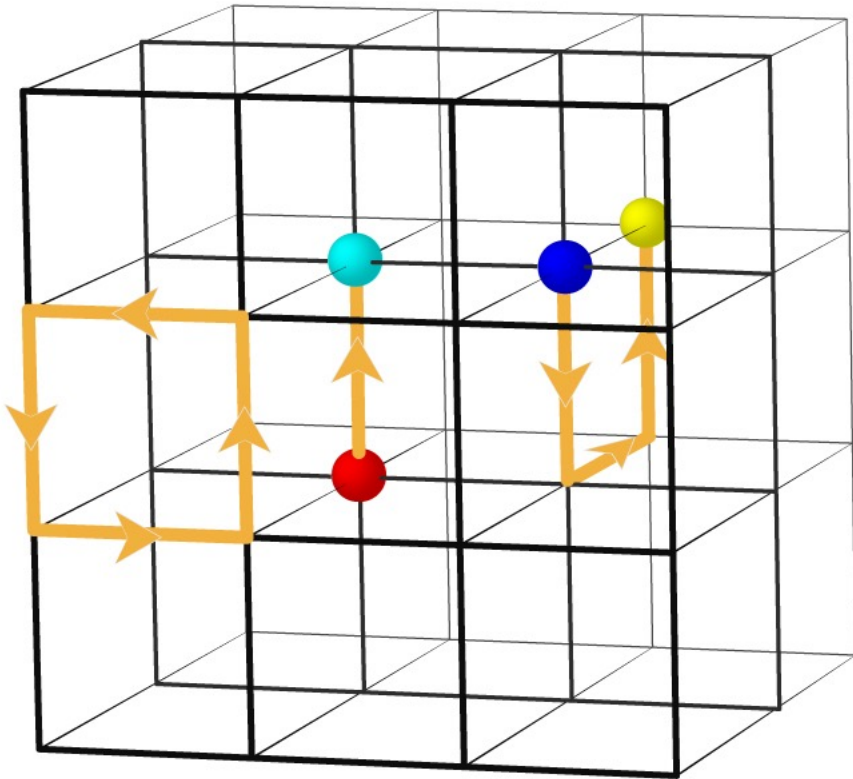


$$\mathcal{L}_{\text{QCD}} = \bar{q}(i\not{D} - m)q - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

でも、QCDのラグランジアンから  
強い力を説明したい！

そうだ、数値計算なら…！

本当は時間1次元+空間3次元



## 格子QCD (lattice QCD)

- 有限体積の箱に入れる
- 時空を離散化 (+ Euclid化)
- クォークをサイトに、グルーオンをリンクに
- 統計的期待値を計算

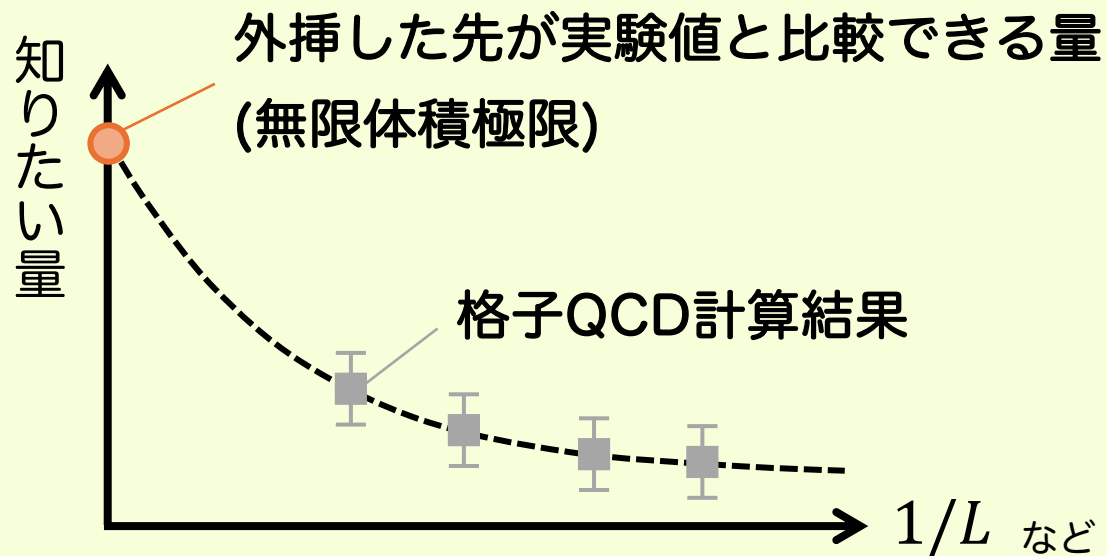
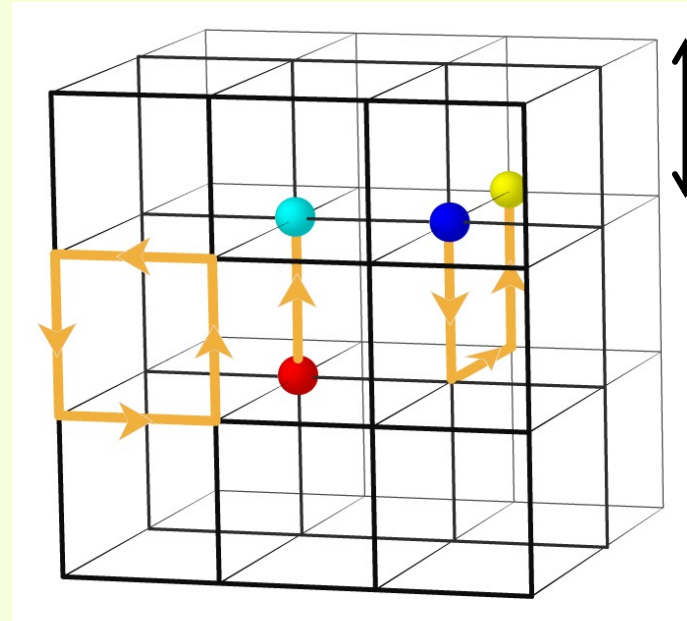
$$S = \int d\tau d^3x \mathcal{L}_{\text{QCD}}$$

確率分布

$$\langle O \rangle = \frac{\int dq dg e^{-S} O}{\int dq dg e^{-S}}$$

## 有限体積効果

- ・ 離散化誤差
- ・ 物理点からずれたパラメーター
- ・ Fit ansatz

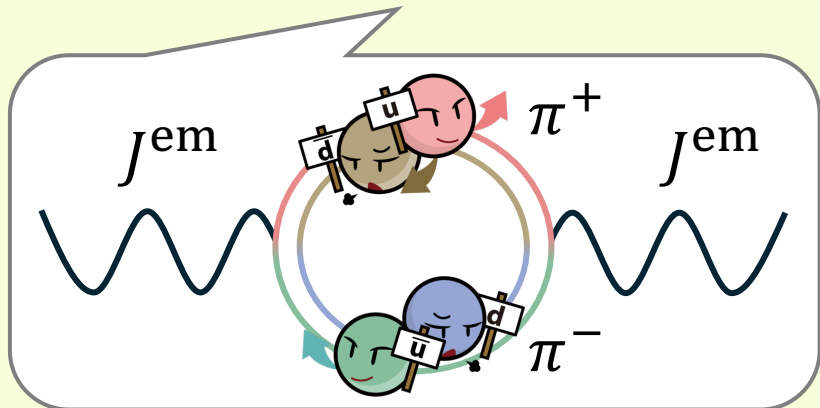


**$n$ 点(相関)関数** = 異なる時空座標  $n$ 点での演算子の相関

$$\langle 0 | \mathcal{O}(x_1) \mathcal{O}(x_2) \cdots \mathcal{O}(x_n) | 0 \rangle = \frac{\int dq dg e^{-S} \mathcal{O}(x_1) \mathcal{O}(x_2) \cdots \mathcal{O}(x_n)}{\int dq dg e^{-S}}$$

例) 電磁カレントの2点関数

$$\langle 0 | J_\mu^{\text{em}\dagger}(x) J_\nu^{\text{em}}(0) | 0 \rangle \xrightarrow{3d \text{ F.T.}} \int d^3x \sum_n \langle 0 | J_\mu^{\text{em}\dagger}(x) | \pi\pi, n \rangle \langle \pi\pi, n | J_\nu^{\text{em}}(0) | 0 \rangle$$



$$= \sum_n |\langle 0 | J^{\text{em}}(0) | \pi\pi, n \rangle|^2_{\mu\nu} e^{-E_{\pi\pi, n} \tau}$$

基本的には、 $n$ 点(相関)関数から求まるもの全てに含まれる！

例えば…

- ・ハドロン  $X$  の (基底状態の) 質量

$$\begin{aligned} \langle 0 | J_X^\dagger(x) J_X(0) | 0 \rangle &\xrightarrow{\text{3d F.T.}} \sum_n |\langle 0 | J_X^\dagger(0) | X, n \rangle|_{\mu\nu}^2 e^{-E_{X,n} \tau} \\ &\xrightarrow{\tau \gg 0} |\langle 0 | J_X^\dagger(0) | X, n=0 \rangle|_{\mu\nu}^2 e^{-E_{X,n=0} \tau} \end{aligned}$$

他にも…

- ・ハドロンの行列要素

次のページから！

- ・ミューオン磁気異常能率のQCDからの寄与



**Muon  $g - 2$**

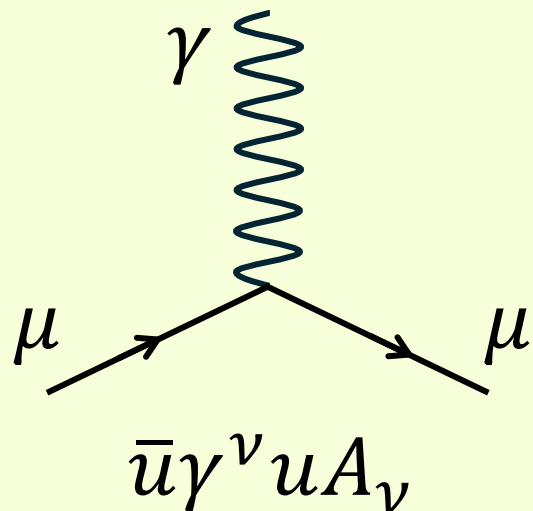
# ランダの $g$ 因子

スピン軌道相互作用

$$\propto g_{\mu} \vec{S} \cdot \vec{B}$$

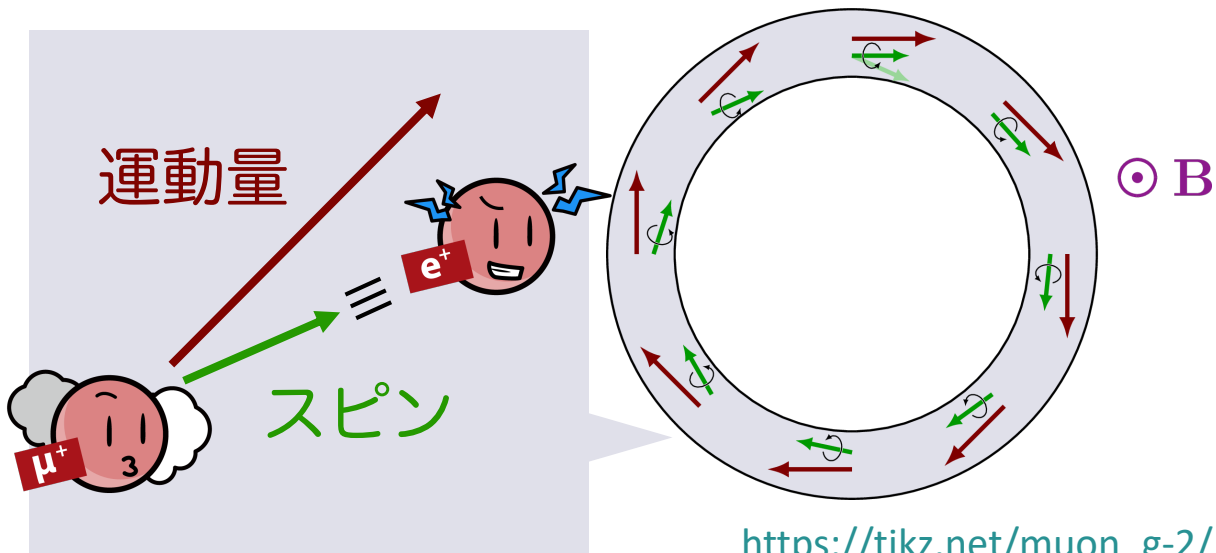
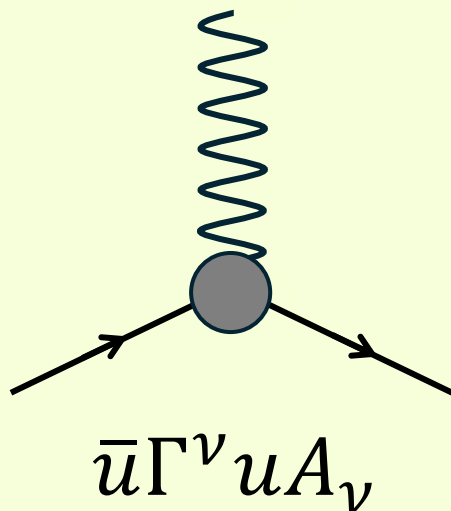
ディラックの理論

$$g_{\mu} = 2$$

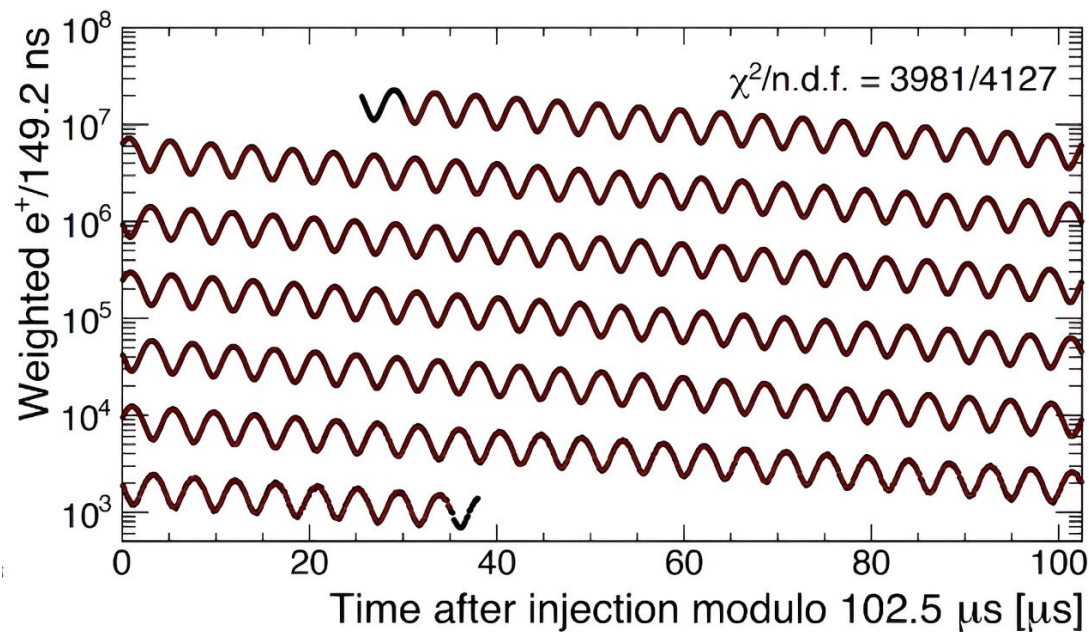


場の理論

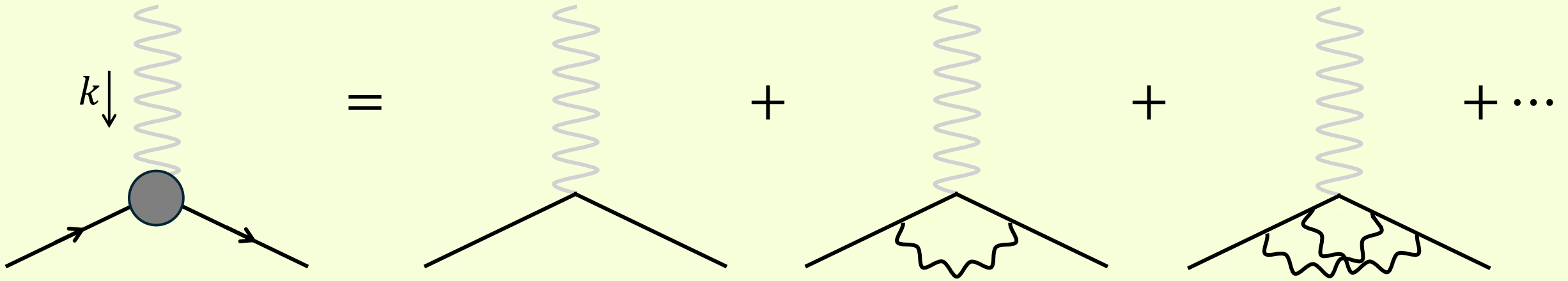
$$g_{\mu} \neq 2$$



[https://tikz.net/muon\\_g-2/](https://tikz.net/muon_g-2/)



[https://phys.org/news/2023-11-muon-g-positive-anomalous-magnetic.html#google\\_vignette](https://phys.org/news/2023-11-muon-g-positive-anomalous-magnetic.html#google_vignette)



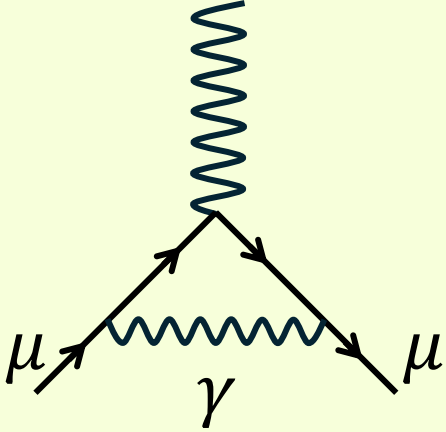
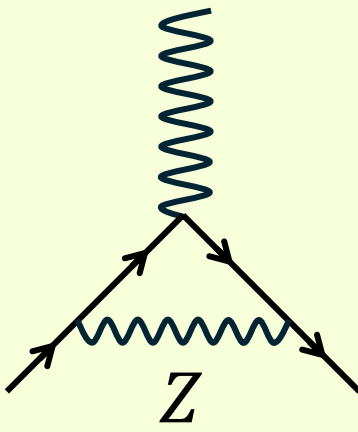
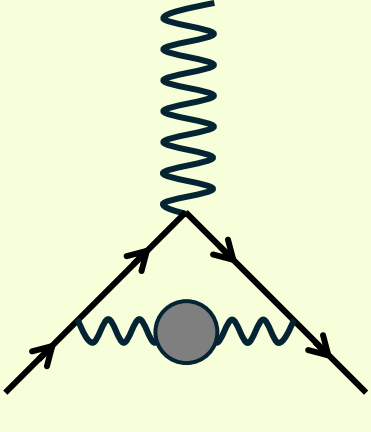
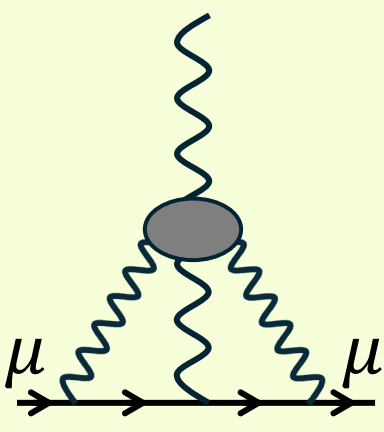
$$\bar{u}\Gamma^\nu(k)u = \bar{u} \left[ \gamma^\nu F_1(k^2) + \frac{i\sigma^{\nu\rho}k_\rho}{2m} F_2(k^2) \right] u$$

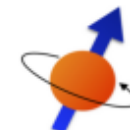
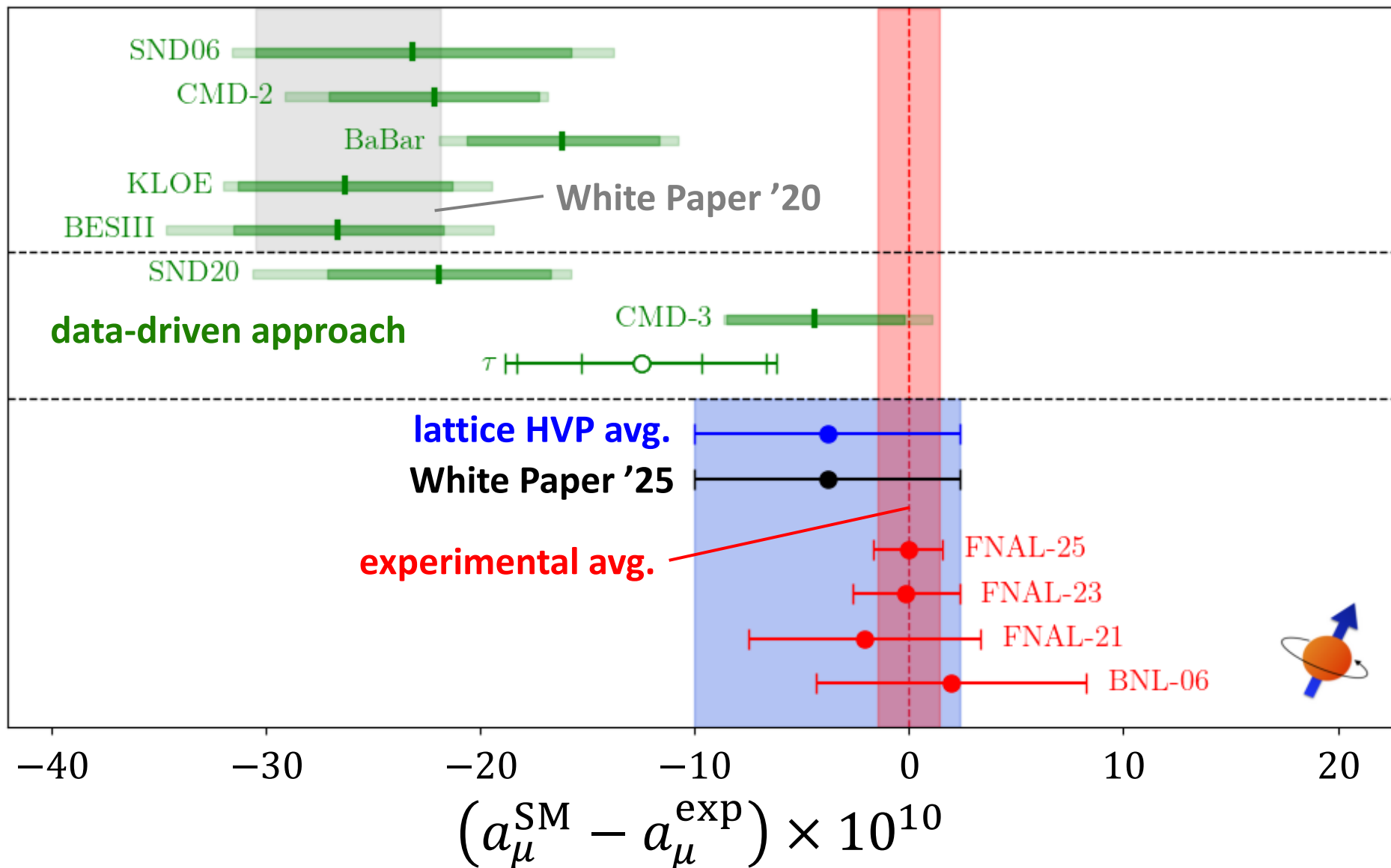
$F_1(0) = 1$  : 電荷の規格化

$$g_\mu = 2[F_1(0) + F_2(0)]$$

ミューオン磁気異常能率

$$a_\mu = F_2(0) = \frac{g_\mu - 2}{2}$$

	QED	electroweak	HVP hadronic vacuum polarization	HLbL hadronic light by light
leading order diagram				
$a_\mu \times 10^{10}$	11,658,471.88(2)	15.44(4)	704.5(6.1)	11.55(99)
寄与 [%]	<b>99.9</b>	0.000132	0.00604	0.0000991
誤差 [%]	0.00105	0.00419	<b>97.4</b>	2.57



ちょっと休憩。

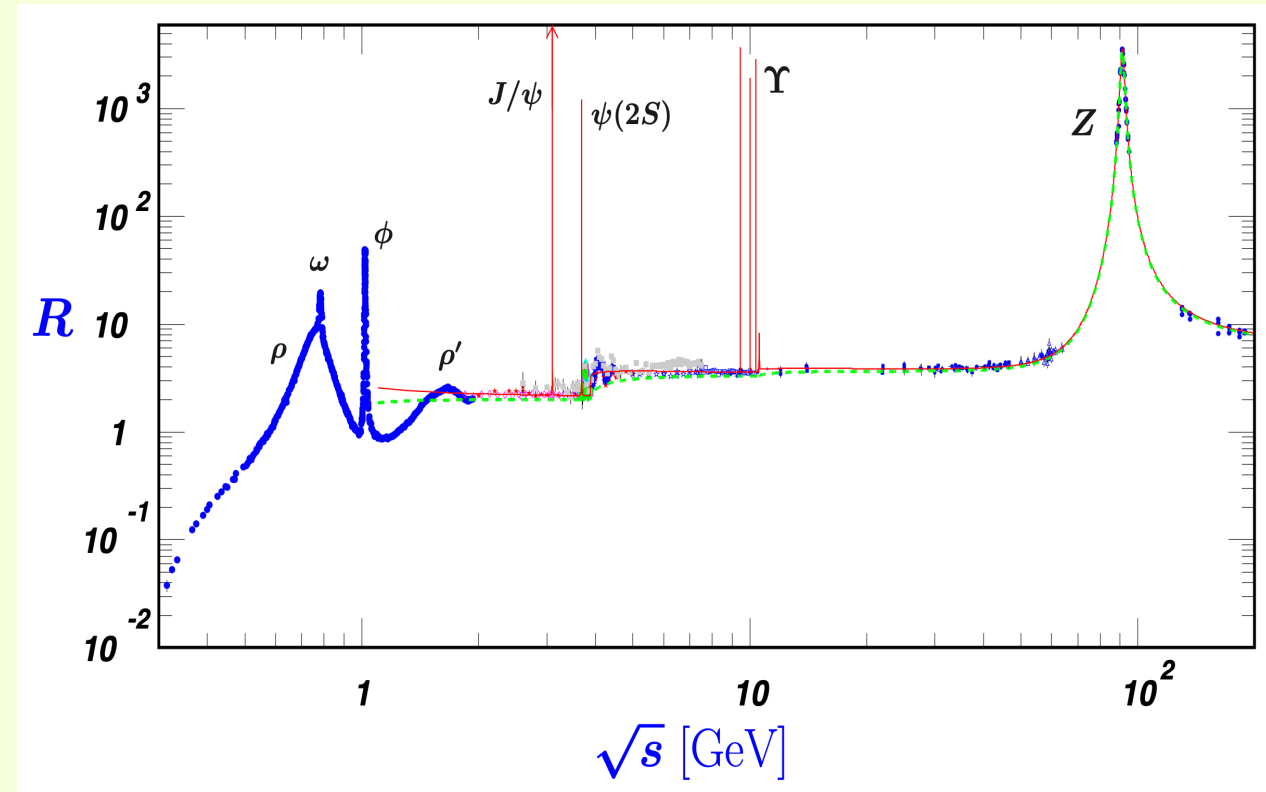
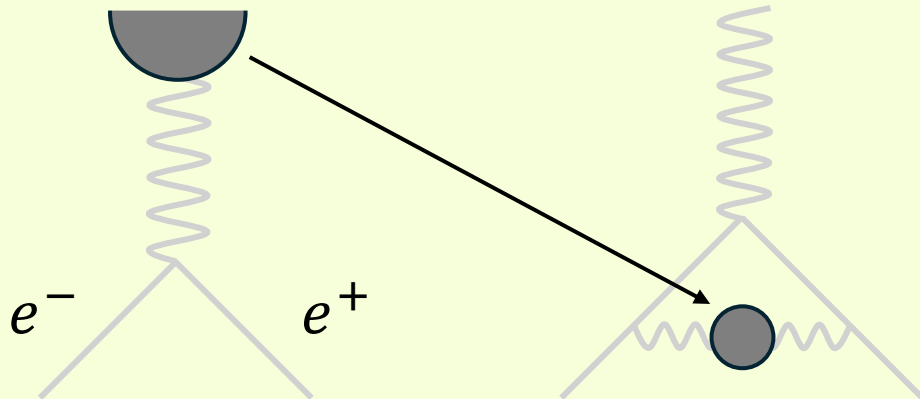
$$a_{\mu}^{\text{HVP,LO}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^{\infty} ds \underbrace{R(s)}_{\text{experiment}} \underbrace{K(s)}_{\text{analytically known}}$$

R ratio from experiment

$$R(s) = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}$$

experiment

HVP in  $g - 2$



## time-momentum representation

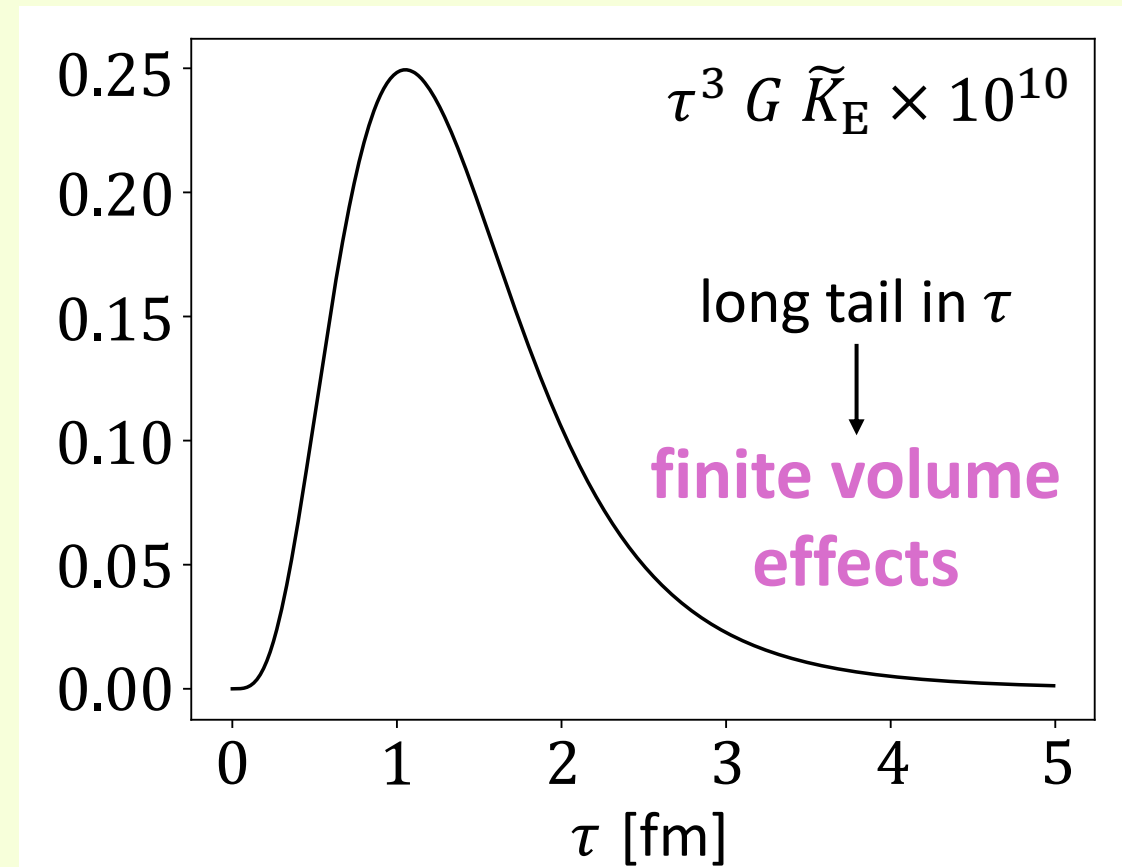
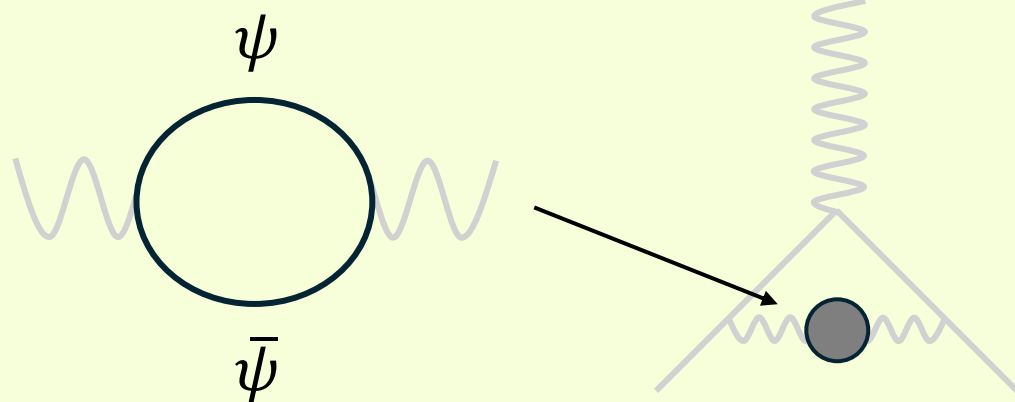
$$a_\mu^{\text{HVP,LO}} = 4\alpha^2 m_\mu \int_0^\infty d\tau \tau^3 \begin{matrix} \text{lattice} \\ G(\tau) \end{matrix} \begin{matrix} \text{analytically known} \\ \tilde{K}_E(\tau) \end{matrix}$$

Euclidean lattice correlation function

$$G(\tau) \equiv \int dx \langle j_z^{\text{em}}(x) j_z^{\text{em}\dagger}(0) \rangle$$

$$j_\nu^{\text{em}} = \bar{\psi} \gamma_\nu \psi$$

HVP in  $g - 2$



equivalent?

$$a_{\mu}^{\text{HVP,LO}} = 4\alpha^2 m_{\mu} \int_0^{\infty} d\tau \tau^3 G(\tau) \tilde{K}_E(\tau)$$

$$a_{\mu}^{\text{HVP,LO}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^{\infty} ds R(s) K(s)$$

$$a_\mu^{\text{HVP,LO}} = 4\alpha^2 m_\mu \int_0^\infty d\tau \tau^3 G(\tau) \tilde{K}_E(\tau)$$

equivalent?

$$a_\mu^{\text{HVP,LO}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty ds R(s) K(s)$$

time-like variable

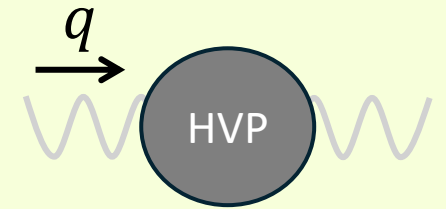
lattice correlator

$$G(\tau) = \langle j_\mu^{\text{em}}(x) j_\nu^{\text{em}}(0) \rangle$$

F.T.  
 $\rightarrow \Pi_{\mu\nu}(Q) = (Q_\mu Q_\nu - \delta_{\mu\nu} Q^2) \Pi(Q^2)$

$$a_\mu^{\text{HVP,LO}} \sim \int_0^\infty dQ^2 [\Pi(Q^2) - \Pi(0)] K_E(Q^2)$$

space-like variable



Minkowski:

$$q^2 = (q_0)^2 - \vec{q}^2$$

Euclid:

$$q_E^2 = -(q_0)^2 - \vec{q}^2 \equiv -Q^2 < 0$$


corresponds to  
the space-like region

equivalent?

time-like variable

$$a_\mu^{\text{HVP,LO}} = 4\alpha^2 m_\mu \int_0^\infty d\tau \tau^3 G(\tau) \tilde{K}_E(\tau)$$

$$a_\mu^{\text{HVP,LO}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty ds R(s) K(s)$$

$$G(\tau) = \langle j_\mu^{\text{em}}(x) j_\nu^{\text{em}}(0) \rangle$$


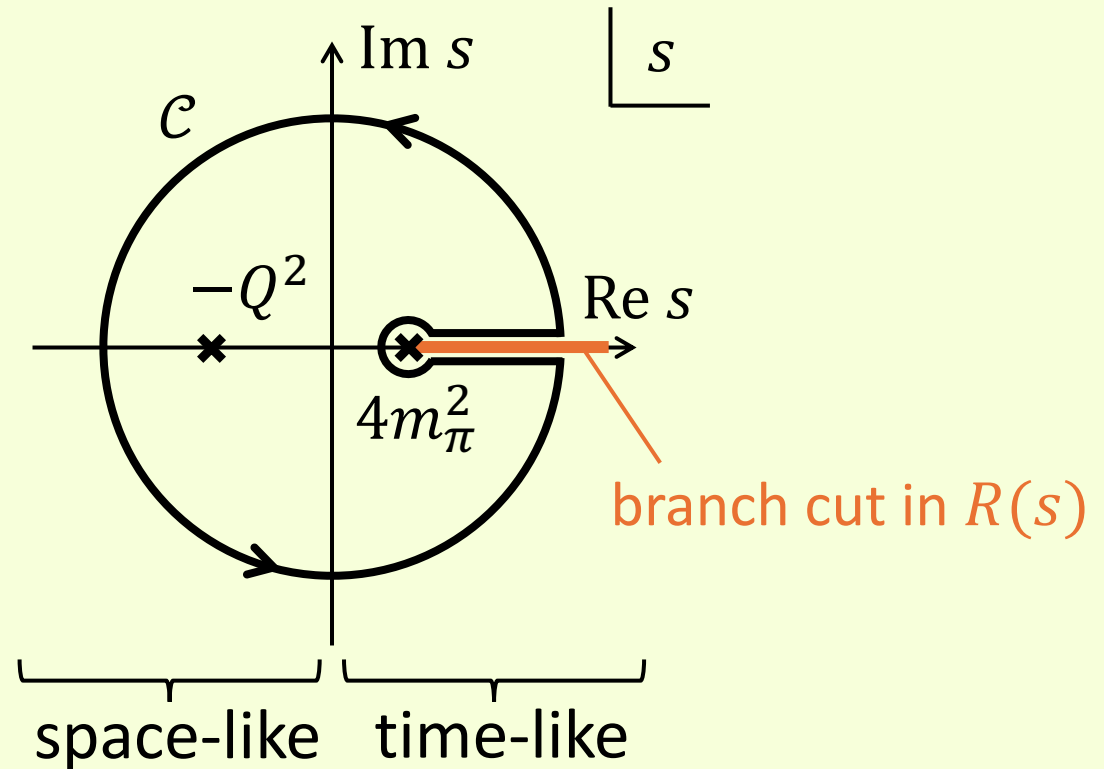
F.T.

$$\rightarrow \Pi_{\mu\nu}(Q) = (Q_\mu Q_\nu - \delta_{\mu\nu} Q^2) \Pi(Q^2)$$

$$a_\mu^{\text{HVP,LO}} \sim \int_0^\infty dQ^2 [\Pi(Q^2) - \Pi(0)] K_E(Q^2)$$

dispersion relation

$$Q^2 \int_{4m_\pi^2}^\infty ds \frac{R(s)}{s(s + Q^2)}$$



# Our research



# construction of $G(\tau, L)$

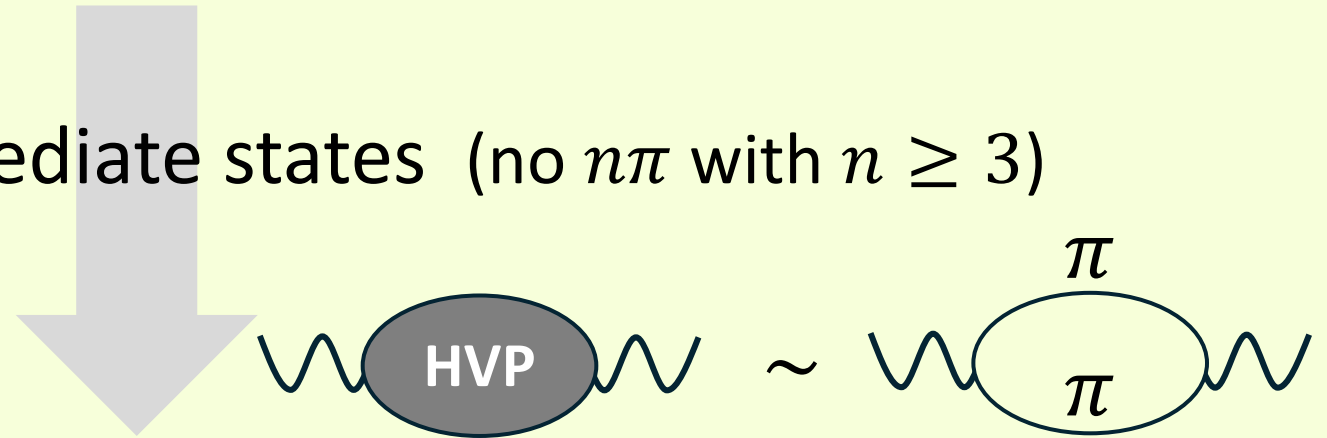
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Euclidean correlator

$$G(\tau) = \int dx^3 \langle 0 | j_Z^{\text{em}}(\tau, \vec{x}) j_Z^{\text{em}}(0) | 0 \rangle$$

assumption:

only  $\pi\pi$  for the intermediate states (no  $n\pi$  with  $n \geq 3$ )



finite V correlator

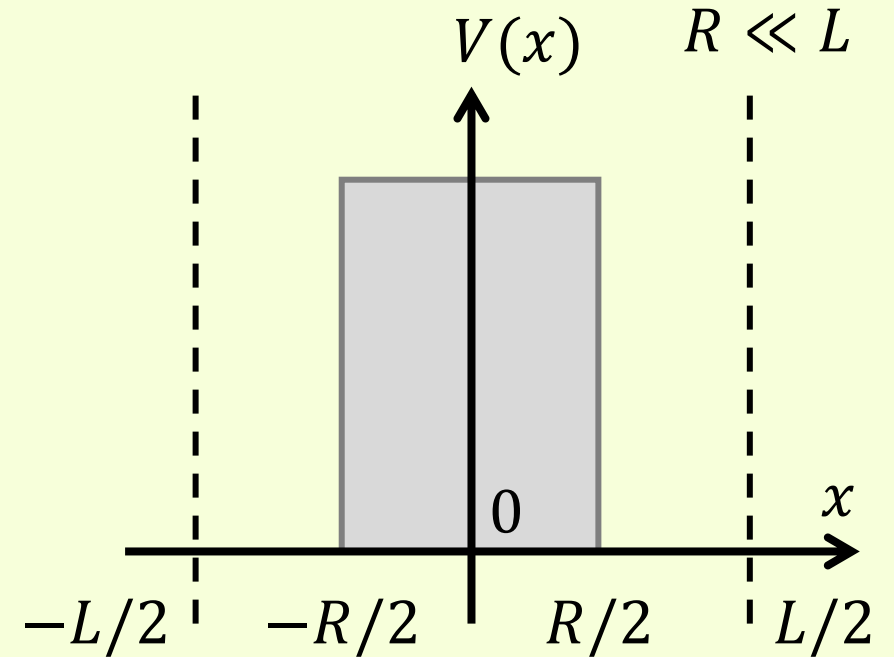
$$G(\tau, L) = \sum_n e^{-E_{\pi\pi, n}\tau} |\langle 0 | j_Z^{\text{em}} | \pi\pi, n \rangle_V|^2$$

**Lüscher formalism**      **Lellouch-Lüscher formula**

wavefunction

$$\psi(x) \sim e^{-ik|x|} + e^{2i\delta(k)} e^{ik|x|}$$

$$(k = \sqrt{2mE}, \delta(k): \text{phase shift})$$



PBC

$$\psi(x - L/2) = \psi(x + L/2)$$

$$e^{2i\delta(k)} e^{ikL} = 1$$

$$2\delta(k) + kL = 2n\pi \quad (n \in \mathbb{Z})$$

free:  $\delta(k) = 0 \implies k = \frac{2n\pi}{L}$

int:  $\delta(k) \neq 0 \implies k = \frac{2\pi}{L} (n + \dots)$

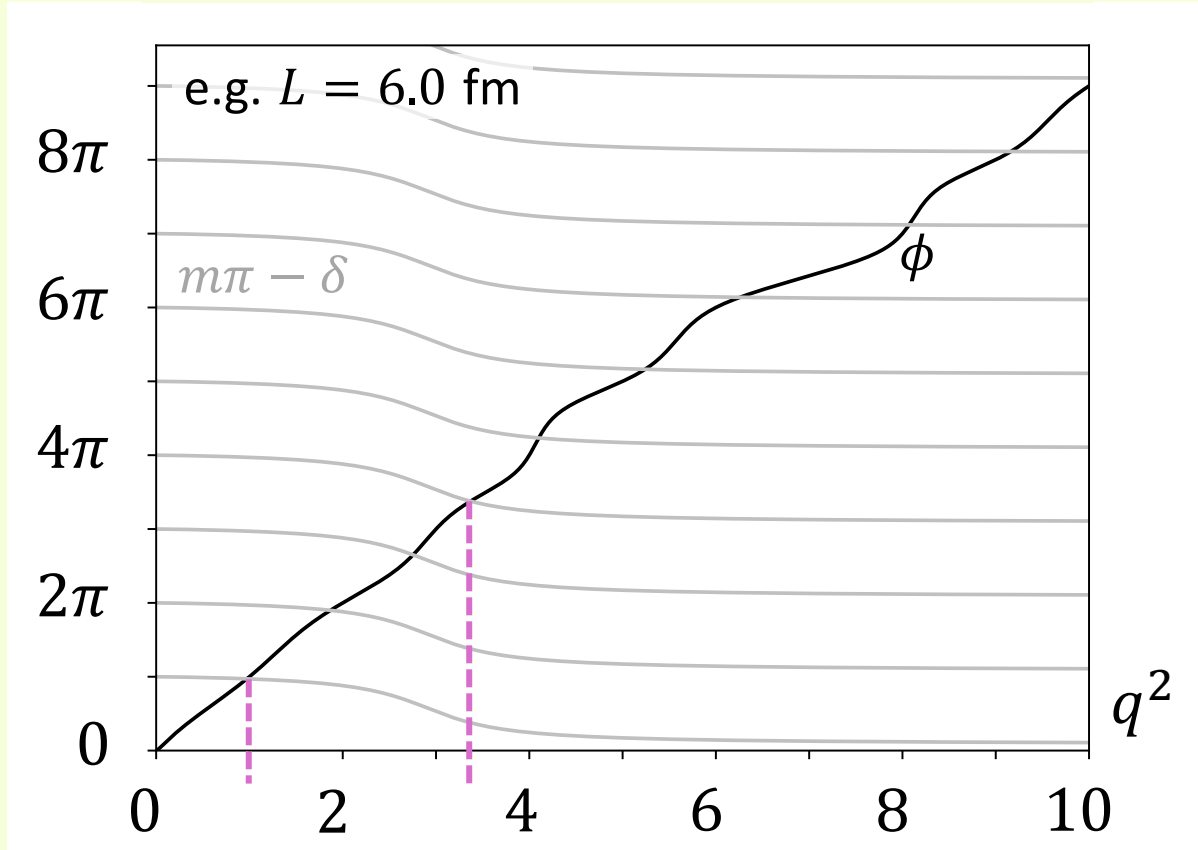
quantization condition

$$\delta(k) + \phi(q) = m\pi \quad (m \in \mathbb{N})$$

$$\tan \phi(q) = -\frac{\pi^{3/2} q}{\mathcal{Z}_{00}(q^2)}, \quad q = \frac{kL}{2\pi}$$

c.f. 1D QM

$$2\delta(k) + kL = 2n\pi \quad (n \in \mathbb{Z})$$



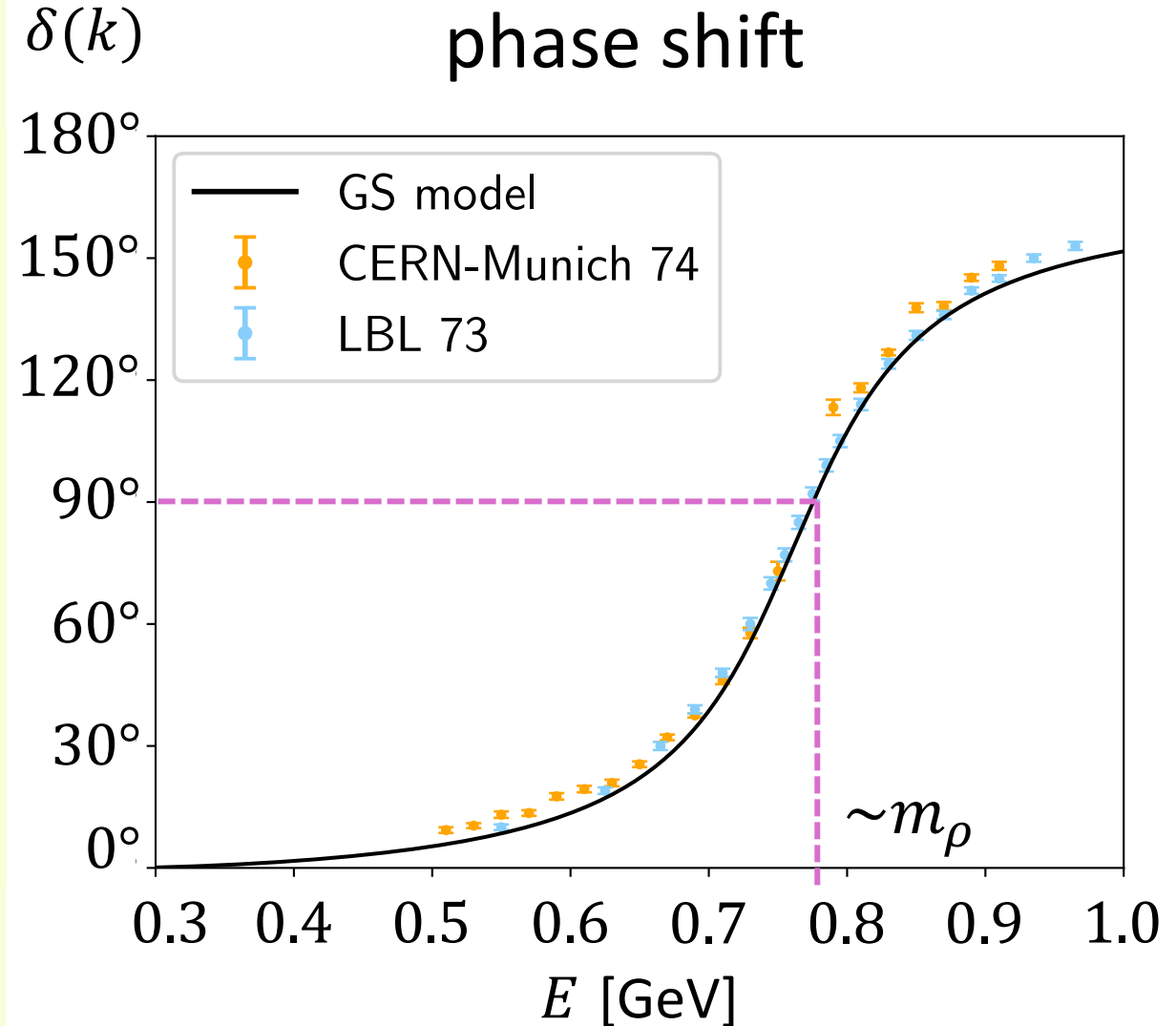
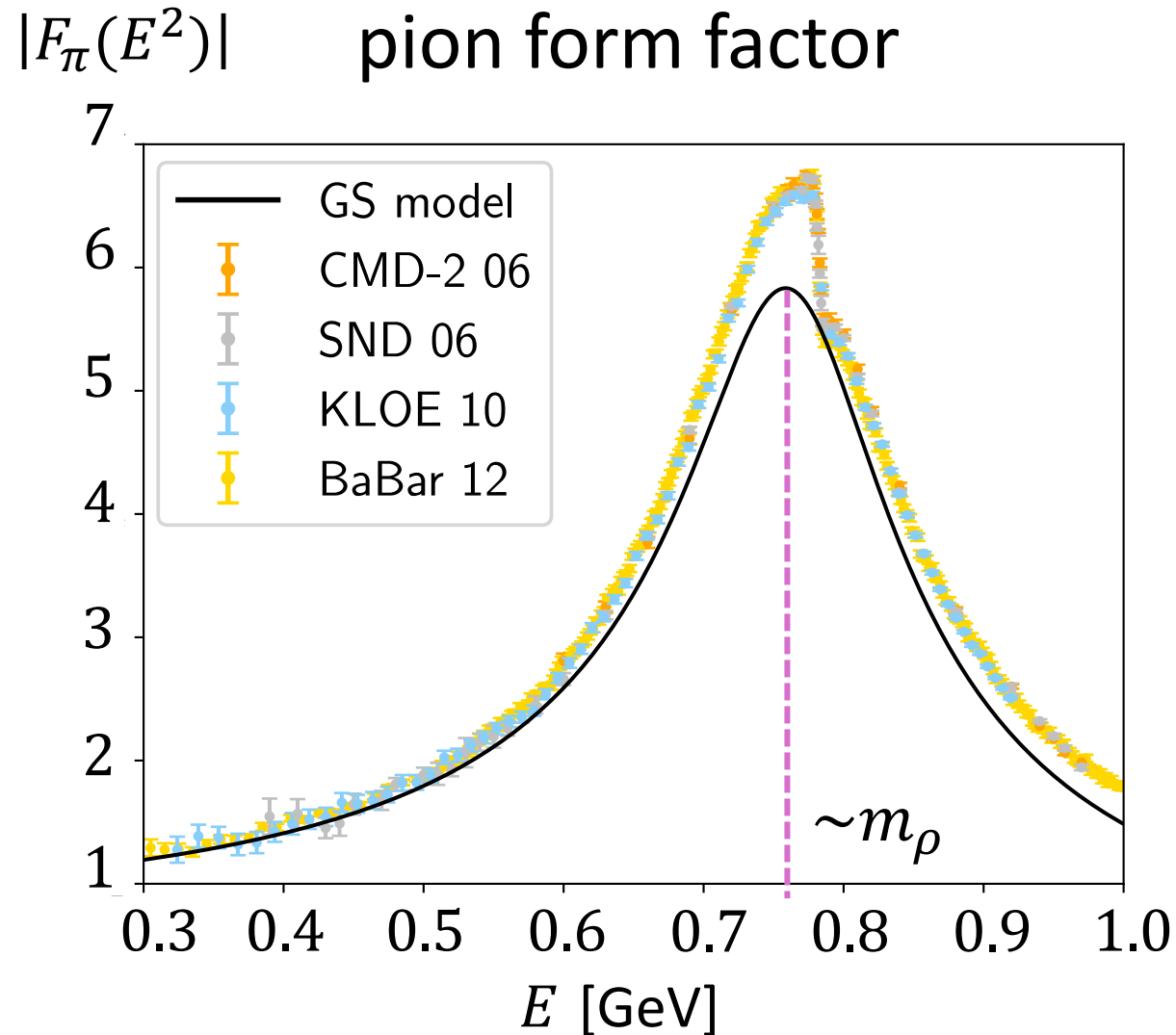
our method:

assume  $\delta(k)$  for  $\pi\pi$  scattering (GS model)  $\longrightarrow$  find the solutions  $q^2$

# Gounaris-Sakurai model

G.J. Gounaris, J.J. Sakurai [PRL 21 (1968)]

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figures from: SI, H. Fukaya, S.Hashimoto [PRD 111 (2025)]

# $\pi\pi$ energy spectrum

figure from: SI, H. Fukaya, S.Hashimoto [PRD 111 (2025)]

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Lüscher's condition

$$\text{int: } \phi(q) = n\pi - \delta^{\text{GS}}(k)$$

$$\text{free: } \phi(q) = n\pi$$

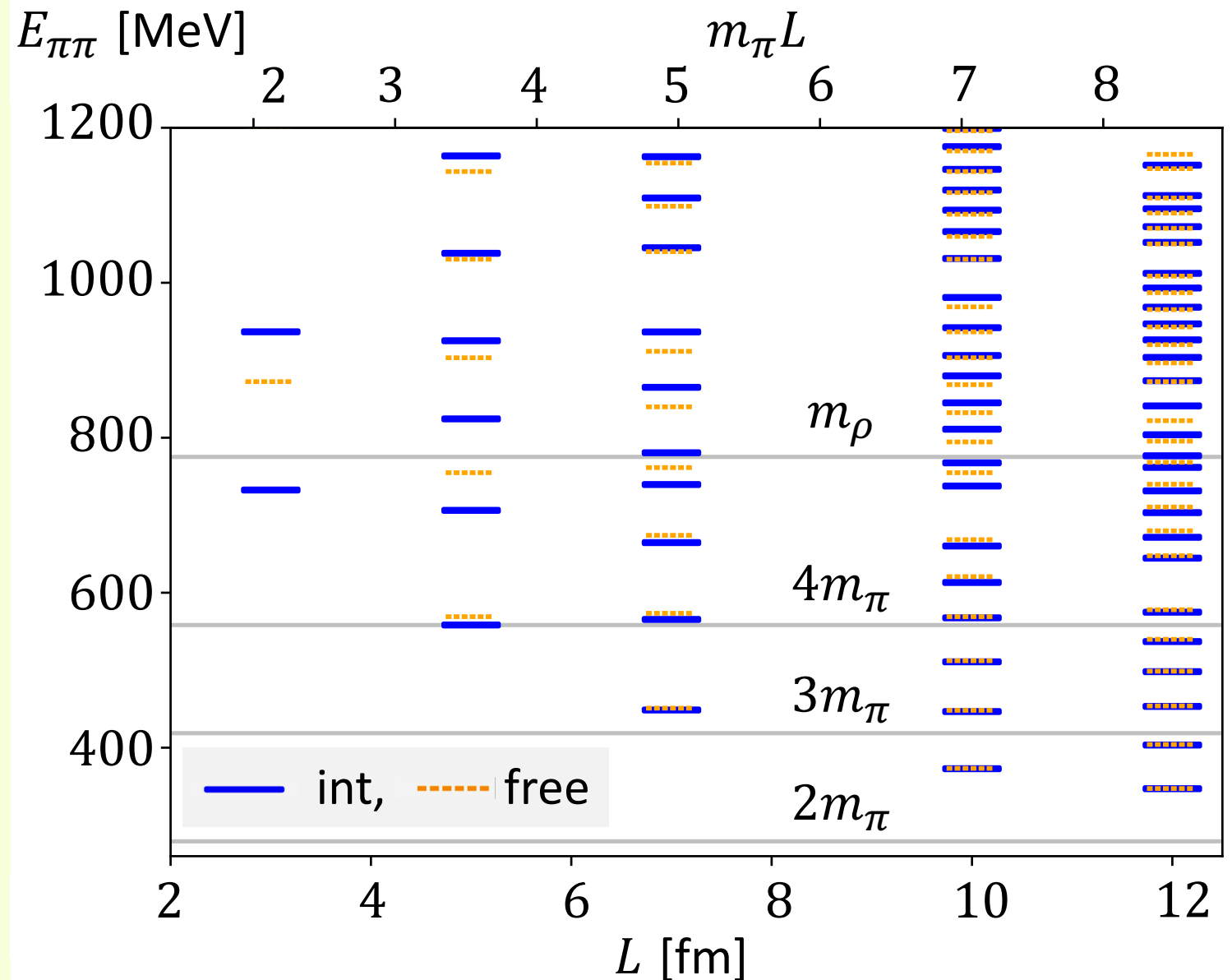
solution

$$q = kL/2\pi$$

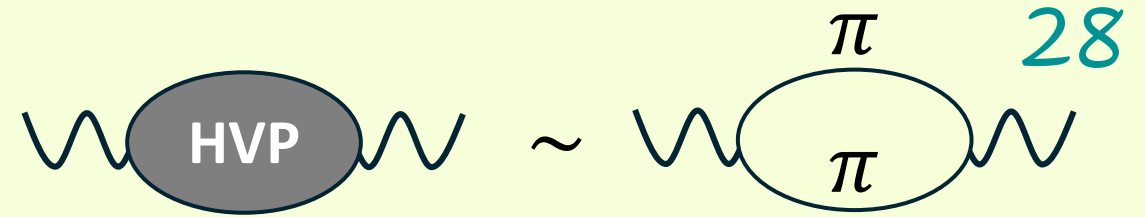
energy spectrum

$$E_{\pi\pi} = 2\sqrt{m_\pi^2 + k^2}$$

( $m_\pi = 139$  MeV; physical  $m_{\pi^\pm}$ )



# construction of $G(\tau, L)$



our assumptions

- only  $\pi\pi$  for the intermediate states (no  $n\pi$  with  $n \geq 3$ )
- the Gounaris-Sakurai model

G.J. Gounaris, J.J. Sakurai [PRL **21** (1968)]

$$G(\tau, L) = \sum_n e^{-E_{\pi\pi, n}\tau} |\langle 0 | j_Z^{\text{em}} | \pi\pi, n \rangle_V|^2$$

## Lüscher formalism

M. Lüscher [Nucl. Phys. B. **354** (1991)]

M. Lüscher [Commun. Math. Phys. **364** (1991)]

$$\phi(q) + \delta^{\text{GS}}(k) = n\pi$$

$$k = 2\pi q/L \rightarrow E_{\pi\pi, n} = 2\sqrt{m_\pi^2 + k^2}$$

## Lellouch-Lüscher formula

H. B. Meyer [PRL **107** (2011)]

$$|F_\pi^{\text{GS}}(E_{\pi\pi, n}^2)|^2$$

$$\propto [q\phi'(q) + k\delta^{\text{GS}'}(k)] |\langle 0 | j_Z^{\text{em}} | \pi\pi, n \rangle_V|^2$$

# FVE on $G(\tau, L)$

$$\Delta G(\tau, L) \equiv G(\tau, L) - G(\tau, \infty)$$

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$\tau = 3.5 \text{ fm}, L \gtrsim 6 \text{ fm}$ :

$$-\Delta G(\tau, L) \sim \left( \frac{4\pi\tau}{m_\pi L^2} \right)^{3/2}$$

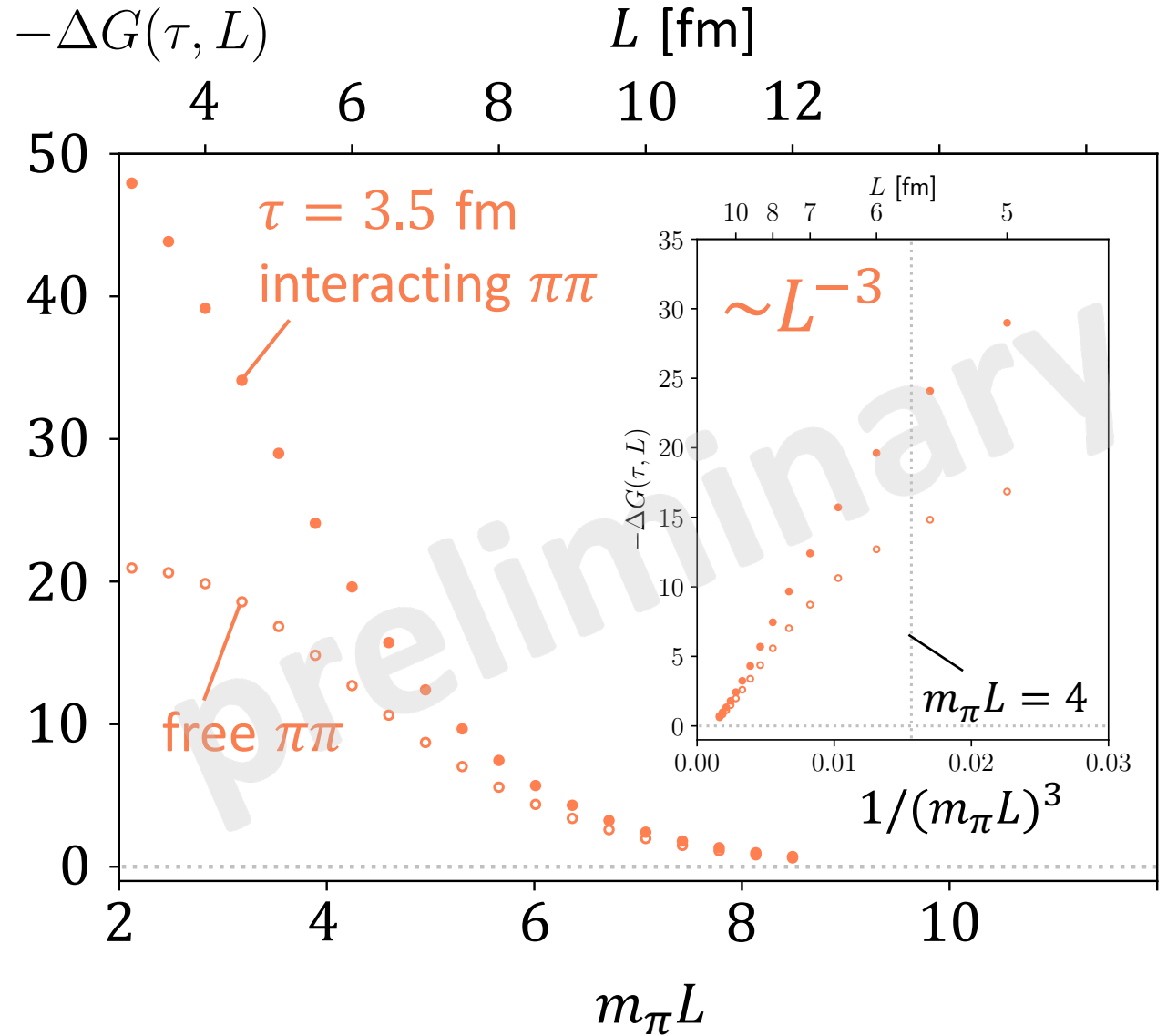
finite-volume:

$$G(\tau, L) = \sum_n e^{-E_{\pi\pi, n} \tau} |\langle 0 | j_Z^{\text{em}} | \pi\pi, n \rangle_V|^2$$

infinite-volume:

$$G(\tau, \infty) = \int_0^\infty dE E^2 \rho(E^2) e^{-E\tau}$$

$$\rho(E^2) = \frac{1}{48\pi^2} \left( 1 - \frac{4m_\pi^2}{E^2} \right)^{3/2} |F_\pi^{\text{GS}}(E^2)|^2$$



# FVE on $G(\tau, L)$

$$\Delta G(\tau, L) \equiv G(\tau, L) - G(\tau, \infty)$$

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$$\tau = L/4 \text{ fm:}$$

$$-\Delta G(\tau, L) \sim e^{-m_\pi L}$$

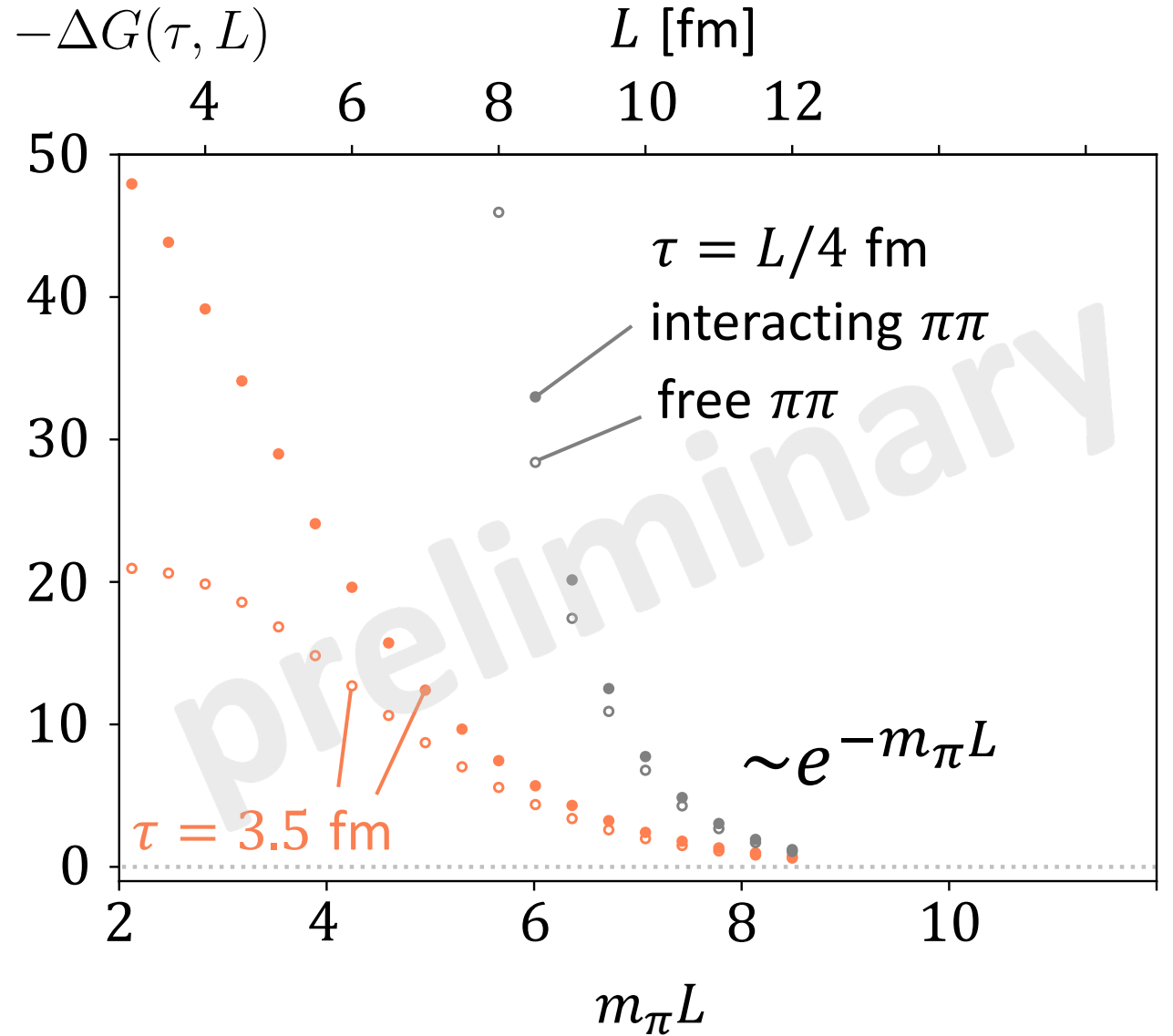
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$$a_\mu^{\text{HVP,LO}}(L) = 4\alpha^2 m_\mu \int_0^{\tau_c} d\tau \tau^3 G(\tau, L) \tilde{K}_E(\tau)$$

finite-volume:

$$G(\tau, L) = \sum_n e^{-E_{\pi\pi, n} \tau} |\langle 0 | j_Z^{\text{em}} | \pi\pi, n \rangle_V|^2$$

infinite-volume:

$$G(\tau, \infty) = \int_0^\infty dE E^2 \rho(E^2) e^{-E\tau}$$

$$\rho(E^2) = \frac{1}{48\pi^2} \left(1 - \frac{4m_\pi^2}{E^2}\right)^{3/2} |F_\pi^{\text{GS}}(E^2)|^2$$

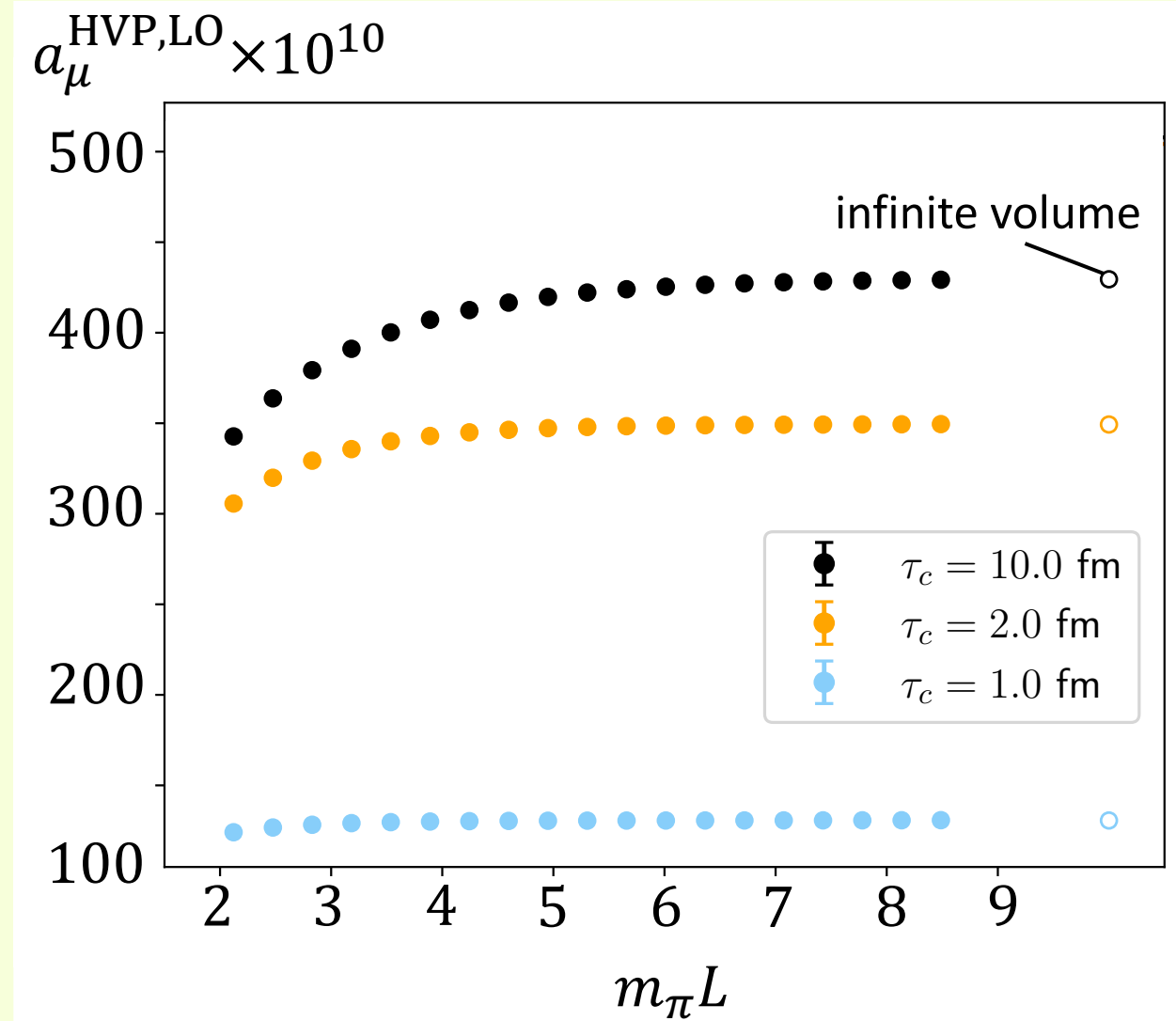


figure from: SI, H. Fukaya, S.Hashimoto [PRD 111 (2025)]

# finite volume effects

$$\Delta a_\mu^{\text{HVP,LO}}(L) \equiv a_\mu^{\text{HVP,LO}}(L) - a_\mu^{\text{HVP,LO}}(\infty) \quad 32$$

$$a_\mu^{\text{HVP,LO}}(L) = 4\alpha^2 m_\mu \int_0^{\tau_c} d\tau \tau^3 G(\tau, L) \tilde{K}_E(\tau)$$

finite-volume:

$$G(\tau, L) = \sum_n e^{-E_{\pi\pi, n} \tau} |\langle 0 | j_Z^{\text{em}} | \pi\pi, n \rangle_V|^2$$

infinite-volume:

$$G(\tau, \infty) = \int_0^\infty dE E^2 \rho(E^2) e^{-E\tau}$$

$$\rho(E^2) = \frac{1}{48\pi^2} \left(1 - \frac{4m_\pi^2}{E^2}\right)^{3/2} |F_\pi^{\text{GS}}(E^2)|^2$$

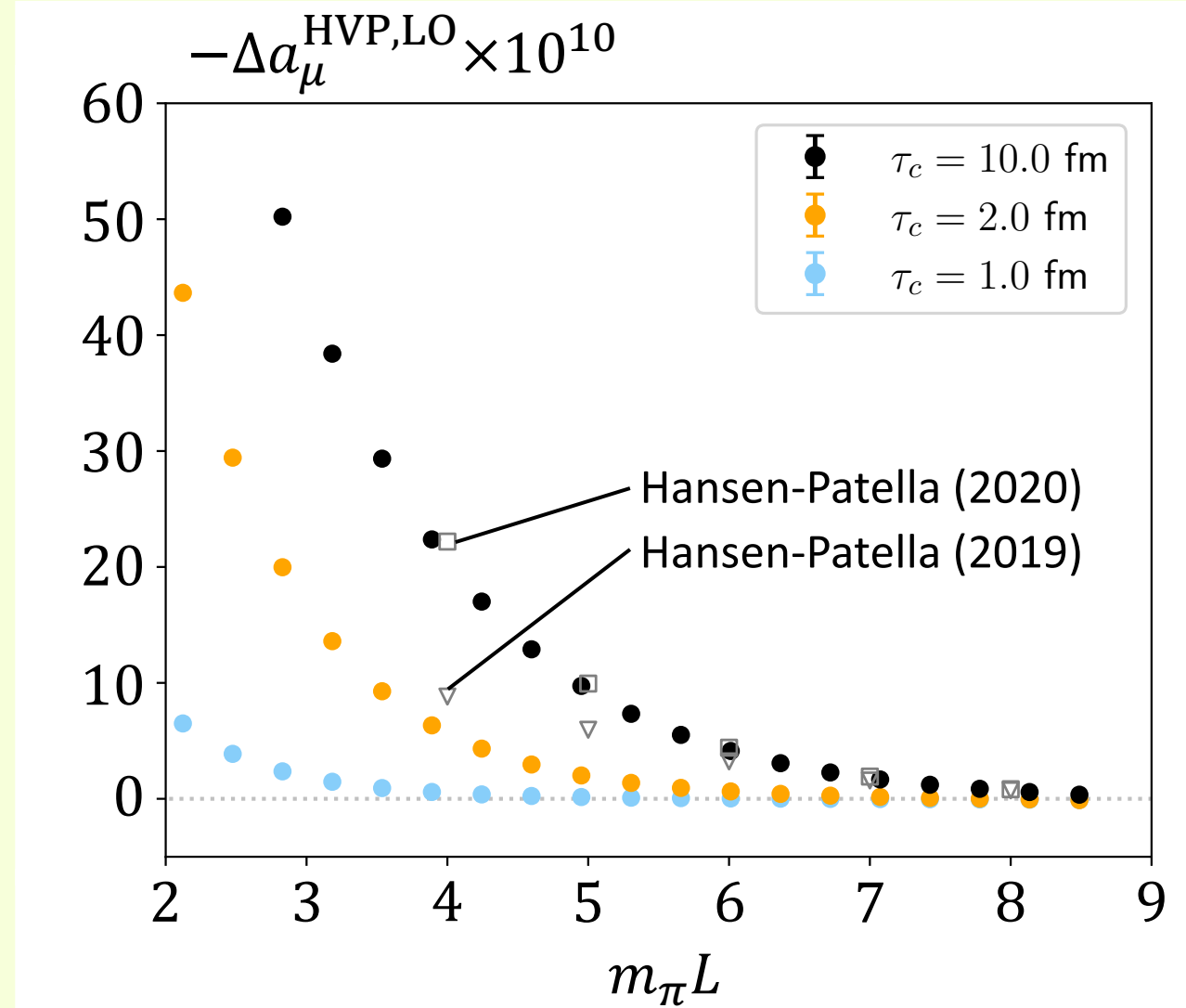


figure from: SI, H. Fukaya, S.Hashimoto [PRD 111 (2025)]

# finite volume effects

$$\Delta a_\mu^{\text{HVP,LO}}(L) \equiv a_\mu^{\text{HVP,LO}}(L) - a_\mu^{\text{HVP,LO}}(\infty) \quad 33$$

c.f.) non-interacting,  $m_\pi \tau \gg 1$

$$\Delta G(\tau, L)$$

$$\supset \sum_{\mathbf{n} \neq 0} \left( 3 - \frac{m_\pi L^2 |\mathbf{n}|^2}{2\tau} \right) \exp\left( -\frac{m_\pi L^2 |\mathbf{n}|^2}{4\tau} \right)$$

A. Francis et. al. [PRD 88 (2013)]

$$= \left( 3 + 2m_\pi \frac{\partial}{\partial m_\pi} \right) \left[ \vartheta_3 \left( 0, e^{-\frac{m_\pi L^2}{4\tau}} \right)^3 - 1 \right] \quad \text{our work}$$

- **large**  $m_\pi L^2 / 4\tau$

**strong exponential suppression**

- **not very large**  $m_\pi L^2 / 4\tau$

**power-law:**  $\vartheta_3 \left( 0, e^{-\frac{m_\pi L^2}{4\tau}} \right)^3 \sim \left( \frac{4\pi\tau}{m_\pi} \right)^{\frac{3}{2}} \frac{1}{L^3}$

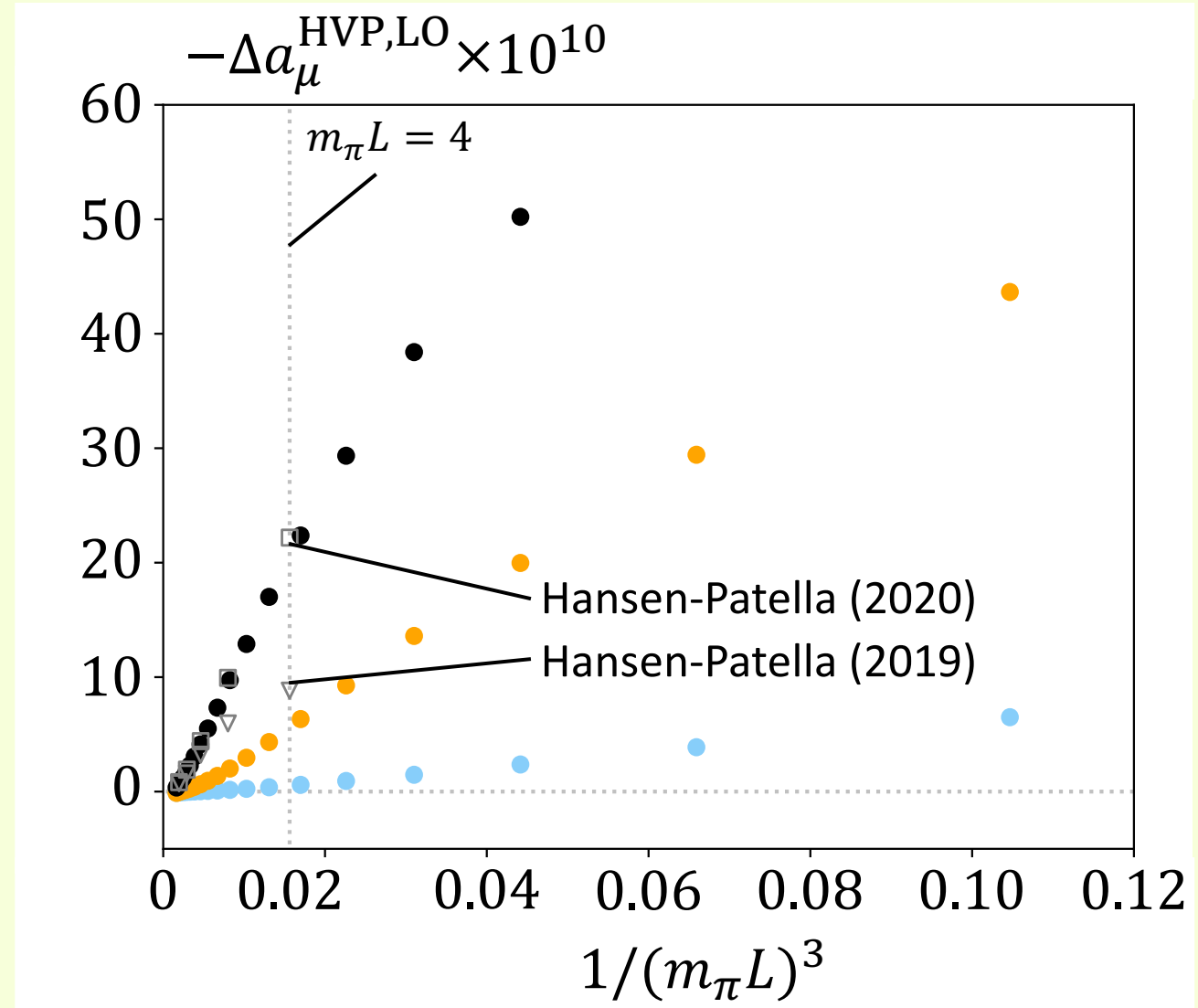


figure from: SI, H. Fukaya, S.Hashimoto [PRD 111 (2025)]

# Summary



- **The two-point function of EM currents has been constructed,** assuming only  $\pi\pi$  and the Gounaris-Sakurai model.
- **Our estimation of  $\Delta a_{\mu}^{\text{HVP, LO}}$  agrees with Hansen-Patella (2020).**  
**A complicated dependence on  $L$  appears.**
- Our method gives a cross-check to LQCD calculation even for large volumes or heavier  $m_{\pi}$ .